MAT0121 - Cálculo Diferencial e Integral II

IAG

Prova 1 - 13/09/2016

Questão	Nota
1	

Nome____

1. Sejam R a região do plano delimitada pelos gráficos de

$$f(x) = \frac{4}{x+1} \text{ e } g(x) = 2\sqrt{x},\ x \in [0,2]$$

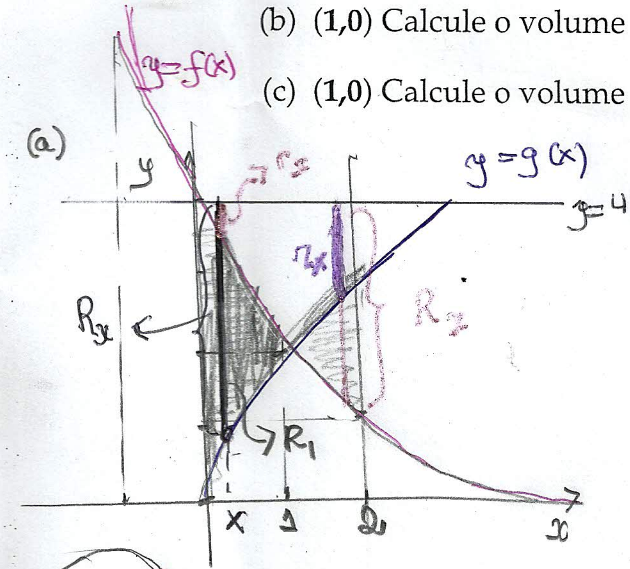
e R_1 a região delimitada pelos gráficos das mesmas funções, mas com $x \in [0,1]$.

(a) **(0,5)** Esboce a região R.

(b) **(1,0)** Calcule o volume do sólido obtido pela rotação de R em torno da reta $y = 4$.

(c) **(1,0)** Calcule o volume so sólido obtido pela rotação de R_1 em torno do eixo y.

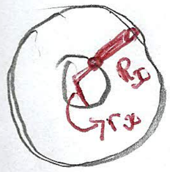
(a)



$f(x) \geq g(x) \iff \frac{2}{x+1} \geq \sqrt{x} \iff$ (ambos positivas)

$\frac{4}{(x+1)^2} \geq x \iff x^3 + 2x^2 + x - 4 \leq 0 \iff$

$(x-1)\underbrace{(x^2+3x+4)}_{>0} \leq 0 \iff x \leq 1$



$$(b)\ V = \pi \int_0^1 \left[(4-2\sqrt{x})^2 - \left(4 - \frac{4}{x+1}\right)^2\right] dx$$

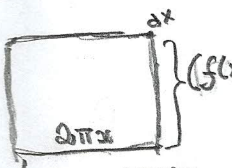
$$+ \pi \int_1^2 \left[\left(4 - \frac{4}{x+1}\right)^2 - (4-2\sqrt{x})^2\right] dx$$

$$= \pi \int_0^1 \left[\frac{32}{x+1} - \frac{16}{(x+1)^2} - 16\sqrt{x} + 4x\right] dx + \pi \int_1^2 \left[-\frac{32}{x+1} + \frac{16}{(x+1)^2} + 16\sqrt{x} - 4x\right] dx$$

$$= \pi \left[32 \ln(x+1)\Big|_0^1 + \frac{16}{x+1}\Big|_0^1 - \frac{32}{3} x^{3/2}\Big|_0^1 + 2x^2\Big|_0^1\right]$$

$$+ \pi \left[-32 \ln(x+1)\Big|_1^2 - \frac{16}{x+1}\Big|_1^2 + \frac{32}{3} x^{3/2}\Big|_1^2 - 2x^2\Big|_1^2\right]$$

Para outra solução de (c) veja a prova B



$$= \pi \left[32 \ln 2 - \frac{50}{3}\right] + \pi \left[32 \ln(2/3) - 14 + \frac{64\sqrt{2}}{3}\right]$$

(c) $V = 2\pi \int_0^1 x\left(\frac{4}{x+1} - 2\sqrt{x}\right) dx = 2\pi \int_0^1 \left[4\left(1 - \frac{1}{x+1}\right) - 2x^{3/2}\right] dx$

Método das Cascas

$$= 2\pi \left[4x\Big|_0^1 - 4\ln(x+1)\Big|_0^1 - \frac{4x^{5/2}}{5}\Big|_0^1\right] = 2\pi\left[\frac{12}{5} - 4\ln 2\right]$$

2. (a) **(1,5)** Calcule
$$\int_4^{+\infty} \frac{1}{x\sqrt{x-4}}\mathrm{d}x.$$

(b) **(1,5)** Use o critério de comparação para determinar se a integral é convergente ou divergente.

i. $\displaystyle\int_e^{+\infty} \frac{\ln x}{\sqrt[5]{x}}\mathrm{d}x$

ii. $\displaystyle\int_2^{4} \frac{\cos^2 x}{\sqrt{x-2}}\mathrm{d}x$

(a) $$\int_4^{+\infty} \frac{dx}{x\sqrt{x-4}} = \underbrace{\int_4^{8} \frac{dx}{x\sqrt{x-4}}}_{(1)} + \underbrace{\int_8^{+\infty} \frac{dx}{x\sqrt{x-4}}}_{(2)}$$

(1) $\displaystyle\int_4^{8} \frac{dx}{x\sqrt{x-4}} = \lim_{t\to 4^+} \int_t^{8} \frac{dx}{x\sqrt{x-4}}$ (2) $\displaystyle\int_8^{+\infty} \frac{dx}{x\sqrt{x-4}} = \lim_{b\to +\infty} \int_8^{b} \frac{dx}{x\sqrt{x-4}}$

Calcular uma primitiva de $\frac{1}{x\sqrt{x-4}}$

$$\int \frac{dx}{x\sqrt{x-4}} = \int \frac{2\cancel{u}\,du}{(u^2+4)\cancel{u}} = \frac{1}{2}\operatorname{arctg}\left(\frac{u}{2}\right) + C = \frac{1}{2}\operatorname{arctg}\left(\frac{\sqrt{x-4}}{2}\right) + C$$

$u = \sqrt{x-4}$
$u^2 = x-4$
$x = u^2+4$
$dx = 2u\,du$

(1) $$\lim_{t\to 4^+}\left[\frac{1}{2}\left[\operatorname{arctg}\left(\frac{\sqrt{8-4}}{2}\right) - \cancel{\operatorname{arctg}\left(\frac{\sqrt{t-4}}{2}\right)}^{\to 0}\right]\right] = \frac{1}{2}\cdot\frac{\pi}{4}$$

(2) $$\lim_{b\to +\infty}\left[\frac{1}{2}\left[\cancel{\operatorname{arctg}\left(\frac{\sqrt{b-4}}{2}\right)}^{\to \pi/2} - \operatorname{arctg}\left(\frac{\sqrt{8-4}}{2}\right)\right]\right] = \frac{\pi}{4} - \frac{\pi}{8}$$

Logo $\displaystyle\int_4^{+\infty} \frac{1}{x\sqrt{x-4}}\,dx = \pi/4$

(b) (i) Se $x \geq e$ então $\ln x \geq 1$. Logo $\frac{\ln x}{\sqrt[5]{x}} \geq \frac{1}{\sqrt[5]{x}}$ (≥ 0)

Mas $$\int_e^{+\infty} \frac{1}{x^{1/5}}\,dx = \lim_{b\to+\infty}\int_e^b x^{-1/5}\,dx = \lim_{b\to+\infty}\left[\frac{5}{4}b^{4/5} - \frac{5}{4}e^{4/5}\right] = +\infty$$

Logo $\int \frac{\ln x}{\sqrt[5]{x}}\,dx$ diverge (pois é maior ou igual a uma que diverge)

(ii) $\forall x,\ 0 \leq \cos^2 x \leq 1$. Logo $0 \leq \frac{\cos^2 x}{\sqrt{x-2}} \leq \frac{1}{\sqrt{x-2}}$.

$$\int_2^4 \frac{1}{\sqrt{x-2}}\,dx = \lim_{t\to 2^+}\int_t^4 (x-2)^{-1/2}\,dx = \lim_{t\to 2^+}\left[2(4-2)^{1/2} - \cancel{2(t-2)^{1/2}}^{\to 0}\right] = 2\sqrt{2} \text{ converge}$$

Logo $\displaystyle\int_2^4 \frac{\cos^2 x}{\sqrt{x-2}}\,dx$ converge

3. Seja $F:\mathbb{R}\to\mathbb{R}$ definida por

$$F(x)=\int_1^{x^3} e^{-t^2}\mathrm{d}t.$$

(a) **(1,0)** Calcule $F'(x)$.

(b) **(1,5)** Use integração por partes para calcular

$$\int_0^1 x^2F(x)\mathrm{d}x.$$

(a) Seja $G(x)$ uma primitiva de e^{-x^2}, isto é, $G'(x)=e^{-x^2}$. Então $F(x)\overset{TFC}{=}G(x^3)-G(1)$.

$$F'(x)=G'(x^3)\,3x^2=e^{-(x^3)^2}3x^2=3e^{-x^6}x^2.$$

(b) $$\int x^2\underbrace{F(x)}_{u}dx=\frac{x^3}{3}F(x)-\int\frac{x^3}{\cancel{3}}\cdot\cancel{3}e^{-x^6}x^2dx$$

$u=F(x)$

$du=F'(x)dx=3e^{-x^6}\cdot x^2$

$dv=x^2dx$

$v=\frac{x^3}{3}$

$$=\frac{x^3}{3}F(x)-\int x^5e^{-x^6}dx$$

$$\int x^5e^{-x^6}dx=-\frac{1}{6}\int e^ydy=-\frac{1}{6}e^y+C=-\frac{1}{6}e^{-x^6}$$

$y=-x^6$

$dy=-6x^5dx$

Logo $$\boxed{\int x^2F(x)dx=F(x)\frac{x^3}{3}+\frac{1}{6}e^{-x^6}+C}$$

Assim $$\int_0^1 x^2F(x)dx=F(x)\frac{x^3}{3}\Big|_0^1+\frac{1}{6}e^{-x^6}\Big|_0^1$$

$$=\frac{F(1)}{3}-\frac{F(0)\cdot 0}{3}+\frac{1}{6}\left(e^{-1}-1\right)$$

$$F(1)=\int_1^1 e^{-t^2}dt=0$$

Portanto $$\int_0^1 x^2F(x)dx=\frac{1}{6}\left(\frac{1}{e}-1\right).$$

4. Seja $\gamma : \mathbb{R} \to \mathbb{R}^2$ definida por $\gamma(t) = (\underbrace{4\,\text{sen}^2 t - 2\cos t + 2}_{x}, \underbrace{1 + 2\cos t}_{y})$.

(a) **(1,0)** Desenhe a imagem de γ.

(b) **(1,0)** Determine a equação da reta tangente à imagem de γ no ponto $(4,2)$.

(a) $y = 1 + 2\cos t \implies \cos t = \frac{y-1}{2}$

$x = 4\,\text{sen}^2 t - 2\cos t + 2 = 4(1-\cos^2 t) - 2\cos t + 2$

$= 4\left(1 - \left(\frac{y-1}{2}\right)^2\right) - (y-1) + 2$

$= 4 - y^2 + 2y \cancel{-1} - y \cancel{+1} + 2 = -y^2 + y + 6$

Assim $\text{Im}\,\gamma \subset \{(x,y) \in \mathbb{R}^2 \mid x = -y^2 + y + 6\}$

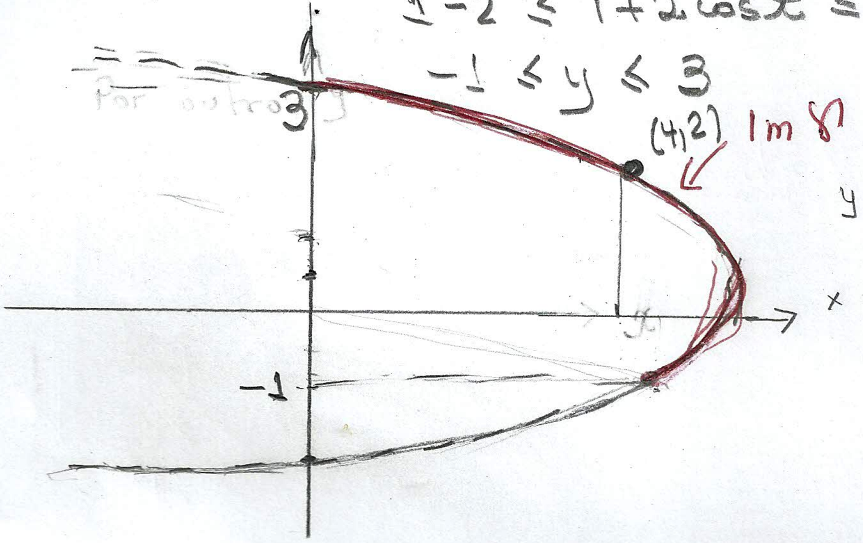
Mas, $\forall\, t \in \mathbb{R}$, vale que

$-1 \leq \cos t \leq 1$

Logo $-2 \leq 2\cos t \leq 2 \implies$

$1 - 2 \leq 1 + 2\cos t \leq 3$

$-1 \leq y \leq 3$



$y = -1$

$x = -1 - 1 + 6$

$-9 + 3 + 6$

(b) $\gamma(t) = (4,2) \implies 1 + 2\cos t = 2 \implies$

$2\cos t = 1 \implies \cos t = \frac{1}{2}$

Se $t = \frac{\pi}{3}$, $4\cdot\left(\frac{\sqrt{3}}{2}\right)^2 - 2\cdot\frac{1}{2} + 2 = 4$

Logo $\gamma(\pi/3) = (4,2)$

$\overrightarrow{\gamma'(t)} = (8\,\text{sen}\,t\cos t + 2\,\text{sen}\,t, -2\,\text{sen}\,t)$

Logo $\overrightarrow{\gamma'(\pi/3)} = \left(8\frac{\sqrt{3}}{2}\cdot\frac{1}{2} + \frac{2\sqrt{3}}{2}, -\frac{2\sqrt{3}}{2}\right) = (3\sqrt{3}, -\sqrt{3})$

$= \sqrt{3}(3,-1)$

Eq. vetorial da reta tangente

$X = (4,2) + \lambda(3,-1), \ \lambda \in \mathbb{R}$

Equação geral: $\langle (1,3), (x-4, y-2) \rangle = 0$

$x - 4 + 3y - 6 = 0$ $\qquad x + 3y - 10 = 0$