# MAT0121 - Cálculo Diferencial e Integral II

## IAG

## Prova 1 - 13/09/2016

1. Sejam $R$ a região do plano delimitada pelos gráficos de

$$f(x) = \frac{2}{x+1} \text{ e } g(x) = \sqrt{x},\ x \in [0,2]$$

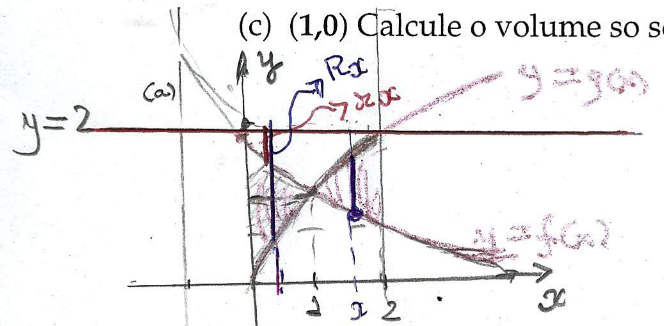
e $R_1$ a região delimitada pelos gráficos das mesmas funções, mas com $x \in [0,1]$.

(a) **(0,5)** Esboce a região $R$.

(b) **(1,0)** Calcule o volume do sólido obtido pela rotação de $R$ em torno da reta $y = 2$.

(c) **(1,0)** Calcule o volume so sólido obtido pela rotação de $R_1$ em torno do eixo $y$.

(a)



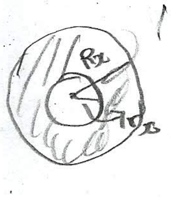

$f(x) \geq g(x) \iff \frac{2}{x+1} \geq \sqrt{x} \iff$ ambas positivas

$\frac{4}{(x+1)^2} \geq x \iff x^3+2x^2+x-4 \leq 0$

$\iff (x-1)\underbrace{(x^2+3x+4)}_{>0} \leq 0 \implies x \leq 1$

(b) $V = \pi \int_0^1 \left[(2-\sqrt{x})^2 - \left(2 - \frac{2}{x+1}\right)^2\right]dx + \pi \int_1^2 \left[\left(2 - \frac{2}{x+1}\right)^2 - (2-\sqrt{x})^2\right]dx$

$= \pi \int_0^1 \left[-4\sqrt{x} + x + \frac{8}{x+1} - \frac{4}{(x+1)^2}\right]dx + \pi \int_1^2 \left[4\sqrt{x} - x - \frac{8}{x+1} + \frac{4}{(x+1)^2}\right]dx$




$= \pi \left[ -\frac{8}{3} x^{3/2}\Big|_0^1 + \frac{x^2}{2}\Big|_0^1 + 8\ln(x+1)\Big|_0^1 + \frac{4}{x+1}\Big|_0^1 \right]$

$+ \pi \left[ \frac{8}{3} x^{3/2}\Big|_1^2 - \frac{x^2}{2}\Big|_1^2 - 8\ln(x+1)\Big|_1^2 - \frac{4}{x+1}\Big|_1^2 \right]$

$= \pi \left[8\ln 2 - 25/6\right] + \pi\left[\frac{16\sqrt{2}}{3} - \frac{7}{2} + 8\ln 2/3\right]$

Para outra solução de (c) veja a prova A!

(c) $V = \pi \int_0^1 y^4\, dy + \pi \int_1^2 \left(\frac{2}{y} - 1\right)^2 dy$

$y = \sqrt{x} \implies x = y^2$

$y = \frac{2}{x+1}$

$y(x+1) = 2 \implies x+1 = 2/y$, $x = 2/y - 1$

$= \pi$

$= \pi \frac{y^5}{5}\Big|_0^1 + \pi\left[-\frac{4}{y}\Big|_1^2 - 4\ln y\Big|_1^2 + y\Big|_1^2\right]$

$= \pi\left[\frac{1}{5} - 2 + 4 - 4\ln 2 + 1\right] = \pi\left[\frac{16}{5} - 4\ln 2\right]$

2. (a) **(1,5)** Calcule
$$\int_9^{+\infty} \frac{1}{x\sqrt{x-9}}\mathrm{d}x.$$

   (b) **(1,5)** Use o critério de comparação para determinar se a integral é convergente ou divergente.

   i. $\int_e^{+\infty} \frac{\ln x}{\sqrt[3]{x}}\mathrm{d}x$

   ii. $\int_3^{5} \frac{\operatorname{sen}^2 x}{\sqrt{x-3}}\mathrm{d}x$

Por definição

$$\int_9^{+\infty} \frac{1}{x\sqrt{x-9}}dx = \int_9^{10} \frac{1}{x\sqrt{x-9}}dx + \int_{10}^{+\infty} \frac{1}{x\sqrt{x-9}}dx$$

$$= \underbrace{\lim_{t\to 9^+} \int_t^{10} \frac{1}{x\sqrt{x-9}}dx}_{(1)} + \underbrace{\lim_{b\to+\infty} \int_{10}^{b} \frac{1}{x\sqrt{x-9}}dx}_{(2)}$$

Calcular $\int \frac{1}{x\sqrt{x-9}}dx$

$u=\sqrt{x-9}$
$u^2 = x-9$
$x = u^2+9$
$dx = 2u\,du$

$$\int \frac{1}{\cancel{u}(u^2+9)} \cdot 2\cancel{u}\,du = \frac{2}{3}\operatorname{arctg}\left(\frac{u}{3}\right)+C$$
$$= \boxed{\frac{2}{3}\operatorname{arctg}\left(\frac{\sqrt{x-9}}{3}\right)+C}$$

(1) $\lim\limits_{t\to 9^+} \left[\frac{2}{3}\operatorname{arctg}\left(\frac{1}{3}\right) - \frac{2}{3}\operatorname{arctg}\left(\frac{\sqrt{t-9}}{3}\right)\right] = \frac{2}{3}\operatorname{arctg}\left(\frac{1}{3}\right)$ $\quad \to 0$

(2) $\lim\limits_{b\to+\infty} \left[\frac{2}{3}\operatorname{arctg}\left(\frac{\sqrt{b-9}}{3}\right) - \frac{2}{3}\operatorname{arctg}\frac{1}{3}\right]$ $\quad \to \frac{2}{3}\cdot\frac{\pi}{2}$

Logo $\int_9^{+\infty} \frac{1}{x\sqrt{x-9}}dx = \pi/3$

(b) (i) Se $x \geq e$ então $\ln x \geq 1$. Logo $\frac{\ln x}{\sqrt[3]{x}} \geq \frac{1}{\sqrt[3]{x}} \geq 0$ $\forall x \geq e$

Mas $\int_e^{+\infty} \frac{1}{\sqrt[3]{x}}dx = \lim\limits_{b\to+\infty} \int_e^b x^{-1/3}dx = \lim\limits_{b\to+\infty} \left[\frac{3}{2}b^{2/3} - \frac{3}{2}e^{2/3}\right] = +\infty$

Assim, $\int_e^{+\infty} \frac{\ln x}{\sqrt[3]{x}}dx$ diverge. (pois é maior ou igual a uma que diverge)

(ii) $0 \leq \operatorname{sen}^2 x \leq 1 \implies 0 \leq \frac{\operatorname{sen}^2 x}{\sqrt{x-3}} \leq \frac{1}{\sqrt{x-3}}$ $\forall x \in ]3,5]$.

$$\int_3^5 \frac{dx}{\sqrt{x-3}} = \lim_{t\to 3^+}\int_t^5 (x-3)^{-1/2}dx = \lim_{t\to 3^+}\left[2(5-3)^{1/2} - 2(t-3)^{1/2}\right] = 2\sqrt{2}$$

Assim $\int_3^5 \frac{dx}{\sqrt{x-3}}$ converge.

3. Seja $F : \mathbb{R} \to \mathbb{R}$ definida por

$$F(x) = \int_1^{x^2} e^{-t^3} dt.$$

(a) **(1,0)** Calcule $F'(x)$.

(b) **(1,5)** Use integração por partes para calcular

$$\int_0^1 x^3 F(x) dx.$$

(a) Seja $G(x)$ uma primitiva qualquer de $e^{-x^3}$, isto é, $G'(x) = e^{-x^3}$.

Então $F(x) = G(x^2) - G(1)$.

Logo $F'(x) = G'(x^2)\, 2x = e^{-(x^2)^3} 2x = 2xe^{-x^6}$

(b) $\int x^3 \underbrace{F(x)}_{u} dx = F(x)\frac{x^4}{4} - \int \frac{x^4}{4} \cdot 2x e^{-x^6} dx$

$u = F(x)$

$du = F'(x)dx = 2xe^{-x^6}$

$dv = x^3 \Rightarrow v = \frac{x^4}{4}$

$\frac{1}{2}\int e^{-x^6} x^5 dx \underset{\substack{y=-x^6 \\ dy = -6x^5 dx}}{=} \frac{-1}{6 \cdot 2} \int e^y dy = -\frac{1}{12} e^y + C = -\frac{1}{12} e^{-x^6} + C$

Logo $\int x^3 F(x) dx = \frac{x^4}{4} F(x) + \frac{1}{12} e^{-x^6} + C$.

Assim, $\int_0^1 x^3 F(x) dx = \frac{x^4}{4} F(x) \Big|_0^1 + \frac{1}{12} e^{-x^6} \Big|_0^1$

$= \frac{1}{4} \underbrace{F(1)}_{\int_1^1 e^{-t^2} dt = 0} - \underbrace{\frac{0}{4} F(0)}_{=0} + \frac{1}{12} e^{-1} - \frac{1}{12} = \frac{1}{12}\left[\frac{1}{e} - 1\right].$

4. Seja $\gamma : \mathbb{R} \to \mathbb{R}^2$ definida por $\gamma(t) = (-4\cos^2 t + 2\,\mathrm{sen}t - 2, 1 + 2\,\mathrm{sen}t)$.

(a) (1,0) Desenhe a imagem de $\gamma$.

(b) (1,0) Determine a equação da reta tangente à imagem de $\gamma$ no ponto $(-4,2)$.

(a) $x = -4\cos^2 t + 2\,\mathrm{sen}t - 2$

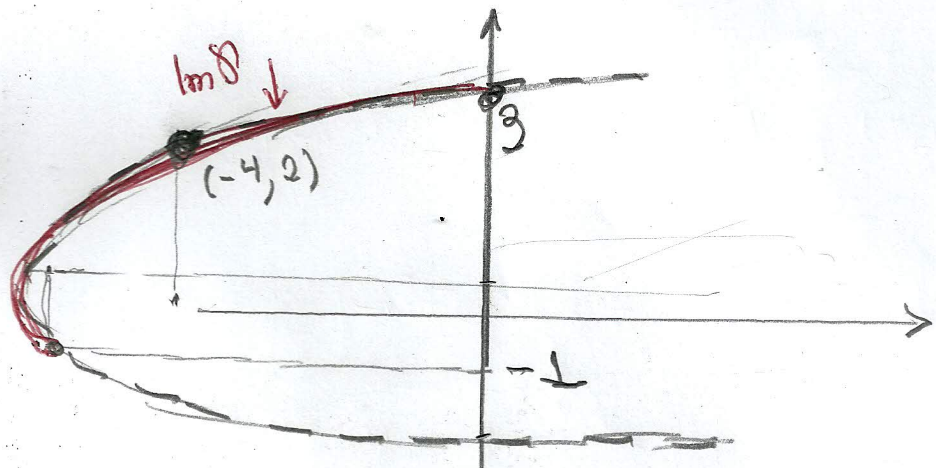
$y = 1 + 2\,\mathrm{sen}t \implies \mathrm{sen}t = \dfrac{y-1}{2}$

$x = -4(1 - \mathrm{sen}^2 t) + 2\,\mathrm{sen}t - 2$

$x = -4 + (y-1)^2 + y - 1 - 2$

$x = y^2 - y - 6$

Assim, $\mathrm{Im}\,\gamma = \{(x,y) \in \mathbb{R}^2 \mid x = y^2 - y - 6\}$




$-1 \leq \mathrm{sen}t \leq 1$

$-2 \leq 2\,\mathrm{sen}t \leq 2$

$1-2 \leq 2\,\mathrm{sen}t + 1 \leq 3$

$-1 \leq y \leq 3$

(b) Encontrar $t$ tal que $\gamma(t) = (-4,2)$

$1 + 2\,\mathrm{sen}t = 2 \implies 2\,\mathrm{sen}t = 1 \quad \mathrm{sen}t = \frac{1}{2}$

Quando $t = \pi/6$, $x(t) = -4\left(\frac{\sqrt{3}}{2}\right)^2 + 1 - 2$

$= -3 - 1 = -4$

Assim $\gamma(\pi/6) = (-4,2)$.

$\gamma'(t) = (-8\cos t(-\mathrm{sen}t) + 2\cos t, 2\cos t)$

$\gamma'(\pi/6) = \left(-\frac{8\sqrt{3}}{2}\left(-\frac{1}{2}\right) + 2\frac{\sqrt{3}}{2}, 2\frac{\sqrt{3}}{2}\right)$

$\gamma'(\pi/6) = (3\sqrt{3}, \sqrt{3}) = \sqrt{3}(3,1)$

Eq. vetorial: $X = (-4,2) + \lambda(3,1)$, $\lambda \in \mathbb{R}$

Eq. geral $\langle (x+4, y-2), (-1, 3) \rangle = 0$

$-x - 4 + 3y - 6 = 0$

$x - 3y + 10 = 0$