MAT0121 - Cálculo Diferencial e Integral II

IAG

Prova 2 - 18/10/2016

Nome____

Número USP:______________________

Assinatura_______________________________________

Questão	Nota
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Total	

1. **(2,0)** Determine o limite, se existir, ou mostre que o limite não existe.

   (a) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{2x^2+3xy^2}{4x^2+3y^4}$

   (b) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^3}{x^4+y^2}\,\text{sen}(x^2+y^4)$

(a) o limite não existe!

Se $\gamma_1(t) = (0,t)$, $\gamma_1(t)$ é contínua em $t=0$ e $\gamma_1(0,0)$

$f(\gamma_1(t)) = \frac{0}{3t^2} = 0 \quad \forall t \neq 0 \quad \Rightarrow \lim_{t\to 0} f(\gamma_1(t)) = \lim_{t\to 0} 0 = 0 = L_1$

Se $\gamma_2(t) = (t^2, t)$, $\gamma_2(t)$ é contínua em $t=0$ e $\gamma_2(0)=(0,0)$.

$f(\gamma_2(t)) = \frac{2t^4+3t^2t^2}{4t^4+3t^4} = \frac{5}{7} \; \forall t\neq 0$. Logo $\lim_{t\to 0} f(\gamma_2(t)) = \frac{5}{7} = L_2$

Como $L_1 \neq L_2$, o limite não existe.

(b) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^3}{x^4+y^2}\,\text{sen}(x^2+y^4) = \lim_{(x,y)\to(0,0)} \underbrace{\frac{x^3}{x^4+y^2}(x^2+y^4)}_{(1)}\;\underbrace{\frac{\text{sen}(x^2+y^4)}{x^2+y^4}}_{(2)}$

(1) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^3(x^2+y^4)}{x^4+y^2} = \lim_{(x,y)\to(0,0)} \underbrace{x}_{\to 0}\underbrace{\frac{x^4}{x^4+y^2}}_{\text{limitada}} + \underbrace{x^3y^2}_{\to 0}\underbrace{\frac{y^2}{x^4+y^2}}_{\text{limitada}} = 0.$

(2) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{\text{sen}(x^2+y^4)}{x^2+y^4} = \lim_{\substack{u = x^2+y^4 \\ u\to 0^+}} \frac{\text{sen}\,u}{u} = 1$

$(x,y) \to (0,0) \Rightarrow u \to 0^+$

Logo $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^3}{x^4+y^2}\,\text{sen}(x^2+y^4) = 0 \cdot 1 = 0.$

2. **(3,5)** Seja $f(x,y) = \dfrac{2x^2 + y^2 - 1}{x^2 - y^2}$.

   (a) Ache e esboce o domínio de f.

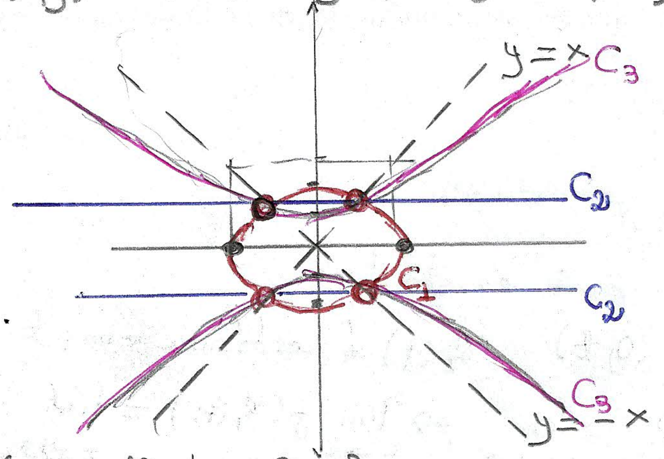
   (b) Esboce as curvas de nível de f nos níveis $k = 1, k = 2$ e $k = 3$.

   (c) Existe $\displaystyle\lim_{(x,y)\to(-\frac{\sqrt{3}}{3},\frac{\sqrt{3}}{3})} f(x,y)$? Explique! (Use (a).)

   (d) Seja $\Gamma :]-\frac{\pi}{2}, \frac{\pi}{2}[\to \mathbb{R}^3$ definida por $\Gamma(t) = (\operatorname{tg} t, \sec t, z(t))$. Sabendo que a imagem de Γ está contida no gráfico de f, determine $z(t)$.

   (e) Determine a equação da reta tangente à trajetória da curva Γ do item (c) no ponto $(1, \sqrt{2}, -3)$.

(a) $D_f = \{(x,y) \in \mathbb{R}^2 \mid x^2 - y^2 \neq 0\} = \{(x,y) \in \mathbb{R}^2 \mid x \neq y \text{ e } x \neq -y\}$



(b) $C_1 = \{(x,y) \in D_f \mid \dfrac{2x^2+y^2-1}{x^2-y^2} = 1\}$ $k = 1$

$2x^2 + y^2 - 1 = x^2 - y^2$

$x^2 + 2y^2 = 1 = x^2 + \dfrac{y^2}{(\sqrt{2}/2)^2} = 1$ elipse

$x = \pm y$

$x^2 + 2x^2 = 1 \Rightarrow x = \pm\dfrac{\sqrt{3}}{3}$

$C_2 = \{(x,y) \in D_f \mid \dfrac{2x^2+y^2-1}{x^2-y^2} = 2\}$ $k = 2$

$\cancel{2x^2} + y^2 - 1 = \cancel{2x^2} - 2y^2$

$3y^2 = 1 \Rightarrow y = \pm\dfrac{\sqrt{3}}{3}$ duas retas horizontais

$C_3 = \{(x,y) \in D_f \mid \dfrac{2x^2+y^2-1}{x^2-y^2} = 3\}$

$2x^2 + y^2 - 1 = 3x^2 - 3y^2$

$4y^2 - x^2 = 1$ hipérbole

(c) O limite não existe! Se $\gamma_1(t) = (t, \sqrt{3}/3)$ (reta $y = \sqrt{3}/3$) então $f(\gamma_1(t)) = 2$ $\forall t \neq \pm\sqrt{3}/3$ (pois essa é a curva de nível 2 de f.)

Logo $\displaystyle\lim_{t\to-\sqrt{3}/3} f(\gamma_1(t)) = 2$. Se "caminharmos" em direção ao ponto $(-\sqrt{3}/3, \sqrt{3}/3)$ pela elipse (C_1), por esse caminho, o limite é 1.

(d) Como $\operatorname{Im}\Gamma \subset G_f$ (*), temos que

$$z(t) = f(\operatorname{tg} t, \sec t) = \frac{2\operatorname{tg}^2 t + \sec^2 t - 1}{\operatorname{tg}^2 t - \sec^2 \theta}$$

$$= \frac{3\operatorname{tg}^2 t}{(-1)} = -3\operatorname{tg}^2 t.$$

(c) $\Gamma(t) = (\operatorname{tg} t, \sec t, -3\operatorname{tg}^2 t)$

$\Gamma(t_0) = (1, \sqrt{2}, -3) \Rightarrow \left.\begin{matrix}\operatorname{tg} t = 1 \\ \sec t = \sqrt{2}\end{matrix}\right\} \Rightarrow t = \pi/4.$

$\vec{\Gamma}'(t) = (\sec^2 t, \sec t \operatorname{tg} t, -6\operatorname{tg} t \sec^2 t)$

$\vec{\Gamma}'(\pi/4) = (2, \sqrt{2}, -6.2)$

$X = (1, \sqrt{2}, -3) + \lambda(2, \sqrt{2}, -12), \lambda \in \mathbb{R}.$

(*) Observe que $\forall t \in \left]-\pi/2, \pi/2\right[$ vale que

$x(t)^2 - y(t)^2 = \operatorname{tg}^2 t - \sec^2 t = -1 \neq 0 \quad \forall t \in \left]-\pi/2, \pi/2\right[.$

Logo $(x(t), y(t)) \in D_f \quad \forall t \in \left]-\pi/2, \pi/2\right[$

3. **(2,0)** Seja $f : \mathbb{R}^2 \to \mathbb{R}$ uma função diferenciável tal que

$$f(t^3 - 2t, 2t^4 - t^2) = t^6 - 2t^2 \quad \text{para todo} \quad t \in \mathbb{R}.$$

(a) Sabendo que $\boxed{\frac{\partial f}{\partial x}(1,1) = \frac{\partial f}{\partial y}(1,1)}$, calcule $\nabla f(1,1)$.

(b) Determine $f(1,1)$ e a equação do plano tangente ao gráfico de f no ponto $(1, 1, f(1,1))$.

(a) $f(\underbrace{t^3-2t}_{x}, \underbrace{2t^4-t^2}_{y}) = t^6 - 2t^2$

$$\frac{d}{dt} f(\underbrace{t^3-2t}_{x}, \underbrace{2t^4-t^2}_{y}) = (t^6-2t^2)'$$

$$\frac{\partial f}{\partial x}(x,y)(3t^2-2) + \frac{\partial f}{\partial y}(x,y)(8t^3-2t) = 6t^5-4t \quad (*)$$

$$(t^3-2t, 2t^4-t^2) = (1,1) \Rightarrow \begin{cases} t^3-2t = 1 \\ 2t^4-t^2 = 1 \end{cases} \Rightarrow \boxed{t=-1}$$

$$t=-1 \Rightarrow (x,y) = (1,1).$$

Substituindo em (*)

$$\frac{\partial f}{\partial x}(1,1)\cdot 1 + \frac{\partial f}{\partial y}(1,1)(-6) = -6+4$$

Por hipótese $\frac{\partial f}{\partial x}(1,1) = \frac{\partial f}{\partial y}(1,1) = a$

Logo $-5a = -2 \Rightarrow a = 2/5$

$$\nabla f(1,1) = (2/5, 2/5)$$

(b) $f(1,1) \overset{t=-1}{=} 1-2 = -1$

$$z = -1 + \frac{2}{5}(x-1) + \frac{2}{5}(y-1)$$

equação do plano tangente

4. **(2,5)** Seja $f(x,y) = \cos\sqrt[3]{x^2+y^2}$.

(a) Calcule $\dfrac{\partial f}{\partial y}(x,y)$ para todo $(x,y) \in \mathbb{R}^2, (x,y) \neq (0,0)$.

(b) Calcule $\dfrac{\partial f}{\partial y}(0,0)$.

(c) Verifique que a função $\dfrac{\partial f}{\partial y}$ é contínua em $(0,0)$.

(d) A função f é diferenciável em $(0,0)$? Por quê?

(a) $\dfrac{\partial f}{\partial y}(x,y) = -\operatorname{sen}\left(\sqrt[3]{x^2+y^2}\right) \cdot \dfrac{1}{3}(x^2+y^2)^{-2/3} \cdot 2y$

$\dfrac{\partial f}{\partial y}(x,y) = -\dfrac{2}{3}\operatorname{sen}\sqrt[3]{x^2+y^2}\,\dfrac{y}{\sqrt[3]{(x^2+y^2)^2}}$ se $(x,y) \neq (0,0)$.

(b) $\dfrac{\partial f}{\partial y}(0,0) = \lim\limits_{y\to 0}\dfrac{f(0,y)-f(0,0)}{y-0} = \lim\limits_{y\to 0}\dfrac{\cos y^{2/3} - 1}{y}$

$\overset{L'H}{\underset{(0/0)}{=}} \lim\limits_{y\to 0}\dfrac{-\operatorname{sen}(y^{2/3})\cdot\frac{2}{3}y^{\frac{2}{3}-1}}{1} = \lim\limits_{y\to 0}\dfrac{-\frac{2}{3}\operatorname{sen}(y^{2/3})}{y^{1/3}\cdot y^{1/3}}\cdot y^{1/3}$

$= \lim\limits_{y\to 0} -\dfrac{2}{3}\underbrace{\dfrac{\operatorname{sen}(y^{2/3})}{y^{2/3}}}_{\to 1}\cdot\underbrace{y^{1/3}}_{\to 0} = \boxed{0}$

(c) $\dfrac{\partial f}{\partial y}(x,y) = \begin{cases} -\dfrac{2}{3}y\,\dfrac{\operatorname{sen}\sqrt[3]{x^2+y^2}}{\sqrt[3]{(x^2+y^2)^2}} & \text{se } (x,y)\neq(0,0) \\ 0 & \text{se } (x,y)=(0,0). \end{cases}$

Temos que mostrar que $\lim\limits_{(x,y)\to(0,0)}\dfrac{\partial f}{\partial y}(x,y) = 0 = \dfrac{\partial f}{\partial y}(0,0)$

$\lim\limits_{(x,y)\to(0,0)}\dfrac{\partial f}{\partial y}(x,y) = \lim\limits_{(x,y)\to(0,0)} -\dfrac{2}{3}y\,\dfrac{\operatorname{sen}\sqrt[3]{x^2+y^2}}{\sqrt[3]{x^2+y}}\cdot\dfrac{1}{\sqrt[3]{x^2+y^2}}$

$= \lim\limits_{(x,y)\to(0,0)} -\dfrac{2}{3}\underbrace{\dfrac{\operatorname{sen}\sqrt[3]{x^2+y^2}}{\sqrt[3]{x^2+y^2}}}_{\to 1}\sqrt[3]{\underbrace{y}_{\to 0}\underbrace{\dfrac{y^2}{x^2+y^2}}_{\text{limitada}}} = 0 = \dfrac{\partial f}{\partial y}(0,0).$

(d) Sim, pois as derivadas parciais $\dfrac{\partial f}{\partial y}$ (e $\dfrac{\partial f}{\partial x}$, por simetria) são contínuas em $(0,0)$.