MAT0121 - Cálculo Diferencial e Integral II

IAG

Prova 1 - 13/09/2016

Questão	Nota
1	

Nome____

1. Sejam R a região do plano delimitada pelos gráficos de

$$f(x) = \frac{4}{x+1} \text{ e } g(x) = 2\sqrt{x},\ x \in [0,2]$$

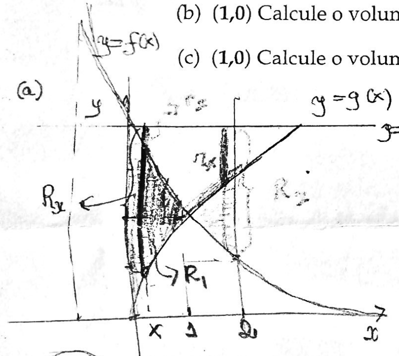
e R_1 a região delimitada pelos gráficos das mesmas funções, mas com $x \in [0,1]$.

(a) (0,5) Esboce a região R.

(b) (1,0) Calcule o volume do sólido obtido pela rotação de R em torno da reta $y = 4$.

(c) (1,0) Calcule o volume so sólido obtido pela rotação de R_1 em torno do eixo y.

(a)



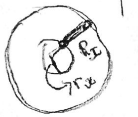
$f(x) \geq g(x) \iff \frac{2}{x+1} \geq \sqrt{x} \iff$ (ambos positivos)

$\frac{4}{(x+1)^2} \geq x \iff x^3 + 2x^2 + x - 4 \leq 0 \iff$

$(x-1)\underbrace{(x^2+3x+4)}_{>0} \leq 0 \iff x \leq 1$

(b) $V = \pi \int_0^1 \left[(4-2\sqrt{x})^2 - \left(4 - \frac{4}{x+1}\right)^2\right] dx$

$+\pi \int_1^2 \left[\left(4 - \frac{4}{x+1}\right)^2 - (4-2\sqrt{x})^2\right] dx$

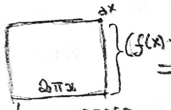


$= \pi \int_0^1 \left[\frac{32}{x+1} - \frac{16}{(x+1)^2} - 16\sqrt{x} + 4x\right] dx + \pi \int_1^2 \left[-\frac{32}{x+1} + \frac{16}{(x+1)^2} + 16\sqrt{x} - 4x\right] dx$

$= \pi \left[32 \ln(x+1)\Big|_0^1 + \frac{16}{x+1}\Big|_0^1 - \frac{32}{3} x^{3/2}\Big|_0^1 + 2x^2\Big|_0^1\right]$

$+ \pi \left[-32\ln(x+1)\Big|_1^2 - \frac{16}{x+1}\Big|_1^2 + \frac{32}{3} x^{3/2}\Big|_1^2 - 2x^2\Big|_1^2\right]$

Para outra solução de (c) veja a prova (B)



$= \pi \left[32 \ln 2 - \frac{50}{3}\right] + \pi \left[32 \ln(2/3) - 14 + \frac{64\sqrt{2}}{3}\right]$

(c) Método das Cascas

$V = 2\pi \int_0^1 x\left(\frac{4}{x+1} - 2\sqrt{x}\right) dx = 2\pi \int_0^1 \left[4\left(\frac{x+1}{x+1} - \frac{1}{x+1}\right) - 2x^{3/2}\right] dx$

$= 2\pi \left[4x\Big|_0^1 - 4\ln(x+1)\Big|_0^1 - \frac{4}{5}x^{5/2}\Big|_0^1\right] = 2\pi\left[\frac{12}{5} - 4\ln 2\right]$

2. (a) **(1,5)** Calcule

$$\int_4^{+\infty} \frac{1}{x\sqrt{x-4}}\,dx.$$

(b) **(1,5)** Use o critério de comparação para determinar se a integral é convergente ou divergente.

i. $\int_e^{+\infty} \frac{\ln x}{\sqrt[5]{x}}\,dx$

ii. $\int_2^{4} \frac{\cos^2 x}{\sqrt{x-2}}\,dx$

(a) $\int_4^{+\infty} \frac{dx}{x\sqrt{x-4}} = \underbrace{\int_4^{8} \frac{dx}{x\sqrt{x-4}}}_{(1)} + \underbrace{\int_8^{+\infty} \frac{dx}{x\sqrt{x-4}}}_{(2)}$

(1) $\int_4^{8} \frac{dx}{x\sqrt{x-4}} = \lim_{t \to 4^+} \int_t^{8} \frac{dx}{x\sqrt{x-4}}$ (2) $\int_8^{+\infty} \frac{dx}{x\sqrt{x-4}} = \lim_{b \to +\infty} \int_8^{b} \frac{dx}{x\sqrt{x-4}}$

Calcular uma primitiva de $\frac{1}{x\sqrt{x-4}}$,

$\int \frac{dx}{x\sqrt{x-4}} = \int \frac{2\cancel{u}\,du}{(u^2+4)\cancel{u}} = \frac{1}{2} \operatorname{arctg}\left(\frac{u}{2}\right) + C = \frac{1}{2} \operatorname{arctg}\left(\frac{\sqrt{x-4}}{2}\right) + C$

$u = \sqrt{x-4}$
$u^2 = x-4$
$x = u^2+4$
$dx = 2u\,du$

(1) $\lim_{t \to 4^+} \left[\frac{1}{2}\left[\operatorname{arctg}\left(\frac{\sqrt{8-4}}{2}\right) - \operatorname{arctg}\left(\frac{\sqrt{t-4}}{2}\right)\right]\right] = \frac{1}{2} \cdot \frac{\pi}{4}$ ($\operatorname{arctg}\left(\frac{\sqrt{t-4}}{2}\right) \to 0$)

(2) $\lim_{b \to +\infty} \left[\frac{1}{2}\left[\operatorname{arctg}\left(\frac{\sqrt{b-4}}{2}\right) - \operatorname{arctg}\left(\frac{\sqrt{8-4}}{2}\right)\right]\right] = \frac{\pi}{4} - \frac{\pi}{8}$ ($\operatorname{arctg}\left(\frac{\sqrt{b-4}}{2}\right) \to \pi/2$)

Logo $\int_4^{+\infty} \frac{1}{x\sqrt{x-4}}\,dx = \pi/4$

(b) (i) Se $x \geq e$ então $\ln x \geq 1$. Logo $\frac{\ln x}{\sqrt[5]{x}} \geq \frac{1}{\sqrt[5]{x}}$ (≥ 0)

Mas $\int_e^{+\infty} \frac{1}{x^{1/5}}\,dx = \lim_{b \to +\infty} \int_e^{b} x^{-1/5}\,dx = \lim_{b \to +\infty} \left[\frac{5}{4} b^{4/5} - \frac{5}{4} e^{4/5}\right] = +\infty$

Logo $\int \frac{\ln x}{\sqrt[5]{x}}\,dx$ diverge (pois é maior ou igual a uma que diverge)

(ii) $\forall x,\ 0 \leq \cos^2 x \leq 1$. Logo $0 \leq \frac{\cos^2 x}{\sqrt{x-2}} \leq \frac{1}{\sqrt{x-2}}$.

$\int_2^4 \frac{1}{\sqrt{x-2}}\,dx = \lim_{t \to 2^+} \int_t^4 (x-2)^{-1/2}\,dx = \lim_{t \to 2^+} \left[2(4-2)^{1/2} - 2(t-2)^{1/2}\right]$ ($(t-2)^{1/2} \to 0$)

$= 2\sqrt{2}$ converge

Logo $\int_2^4 \frac{\cos^2 x}{\sqrt{x-2}}\,dx$ converge.

4. **(2,5)** Seja $f(x,y) = \cos\sqrt[3]{x^2+y^2}$.

(a) Calcule $\frac{\partial f}{\partial y}(x,y)$ para todo $(x,y) \in \mathbb{R}^2, (x,y) \neq (0,0)$.

(b) Calcule $\frac{\partial f}{\partial y}(0,0)$.

(c) Verifique que a função $\frac{\partial f}{\partial y}$ é contínua em $(0,0)$.

(d) A função f é diferenciável em $(0,0)$? Por quê?

(a) $\frac{\partial f}{\partial y}(x,y) = -\operatorname{sen}\left(\sqrt[3]{x^2+y^2}\right) \cdot \frac{1}{3}(x^2+y^2)^{-2/3} \cdot 2y$

$\frac{\partial f}{\partial y}(x,y) = -\frac{2}{3} \operatorname{sen}\sqrt[3]{x^2+y^2}\, \frac{y}{\sqrt[3]{(x^2+y^2)^2}}$ se $(x,y) \neq (0,0)$.

(b) $\frac{\partial f}{\partial y}(0,0) = \lim_{y\to 0} \frac{f(0,y)-f(0,0)}{y-0} = \lim_{y\to 0} \frac{\cos y^{2/3} - 1}{y}$

$\overset{\text{L'H}}{\underset{(0/0)}{=}} \lim_{y\to 0} \frac{-\operatorname{sen}(y^{2/3}) \cdot \frac{2}{3} y^{\frac{2}{3}-1}}{1} = \lim_{y\to 0} \frac{-\frac{2}{3}\operatorname{sen}(y^{2/3})}{y^{1/3}\cdot y^{1/3}} \cdot y^{1/3}$

$= \lim_{y\to 0} -\frac{2}{3} \underbrace{\frac{\operatorname{sen}(y^{2/3})}{y^{2/3}}}_{\to 1} \cdot \underbrace{y^{1/3}}_{\to 0} \boxed{= 0}$

(c) $\frac{\partial f}{\partial y}(x,y) = \begin{cases} -\frac{2}{3} y \frac{\operatorname{sen}\sqrt[3]{x^2+y^2}}{\sqrt[3]{(x^2+y^2)^2}} & \text{se } (x,y) \neq (0,0) \\ 0 & \text{se } (x,y) = (0,0). \end{cases}$

Temos que mostrar que $\lim_{(x,y)\to(0,0)} \frac{\partial f}{\partial y}(x,y) = 0 = \frac{\partial f}{\partial y}(0,0)$

$\lim_{(x,y)\to(0,0)} \frac{\partial f}{\partial y}(x,y) = \lim_{(x,y)\to(0,0)} -\frac{2}{3} y \frac{\operatorname{sen}\sqrt[3]{x^2+y^2}}{\sqrt[3]{x^2+y}} \cdot \frac{1}{\sqrt[3]{x^2+y^2}}$

$= \lim_{(x,y)\to(0,0)} -\frac{2}{3} \underbrace{\frac{\operatorname{sen}\sqrt[3]{x^2+y^2}}{\sqrt[3]{x^2+y^2}}}_{\to 1} \sqrt[3]{\underbrace{y}_{\to 0} \underbrace{\frac{y^2}{x^2+y^2}}_{\text{limitada}}} = 0 = \frac{\partial f}{\partial y}(0,0).$

(d) Sim, pois as derivadas parciais $\frac{\partial f}{\partial y}$ (e $\frac{\partial f}{\partial x}$, por simetria) são contínuas em $(0,0)$.