MAT2351 - CÁLCULO I - VÁRIAS VARIÁVEIS

LICENCIATURA - NOTURNO FÍSICA

1O SEM 2019 - PROFA. DANIELA M. VIEIRA

PRIMEIRA PROVA - 03/04/2019

EM TODAS AS QUESTÕES, JUSTIFIQUE SUA RESPOSTA!

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Questão 1 (2,5)

(a) (1,5) Esboce a curvas polares $r_1 = 3\sin\theta$ e $r_2 = 1 + \sin\theta$, para $0 \le \theta \le 2\pi$.

(b) (1,0) Determine a área da região que está dentro da curva r_1 e fora da curva r_2.

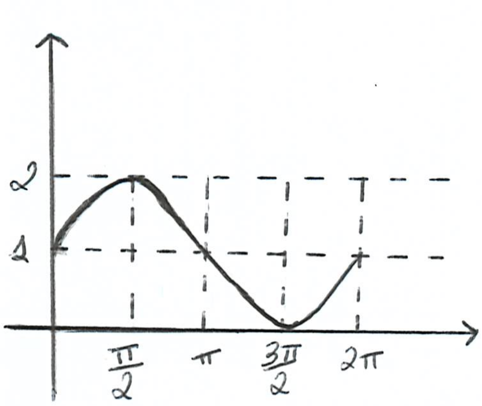
a)

$$r_1 = 3\,\text{sen}\,\theta \Rightarrow \begin{cases} x = 3\,\text{sen}\,\theta\cos\theta = \frac{3}{2}\,\text{sen}(2\theta) \\ y = 3\,\text{sen}\,\theta\,\text{sen}\,\theta = 3\,\text{sen}^2\theta = \frac{3}{2}(1-\cos(2\theta)) = \frac{3}{2} - \frac{3}{2}\cos(2\theta) \end{cases}$$

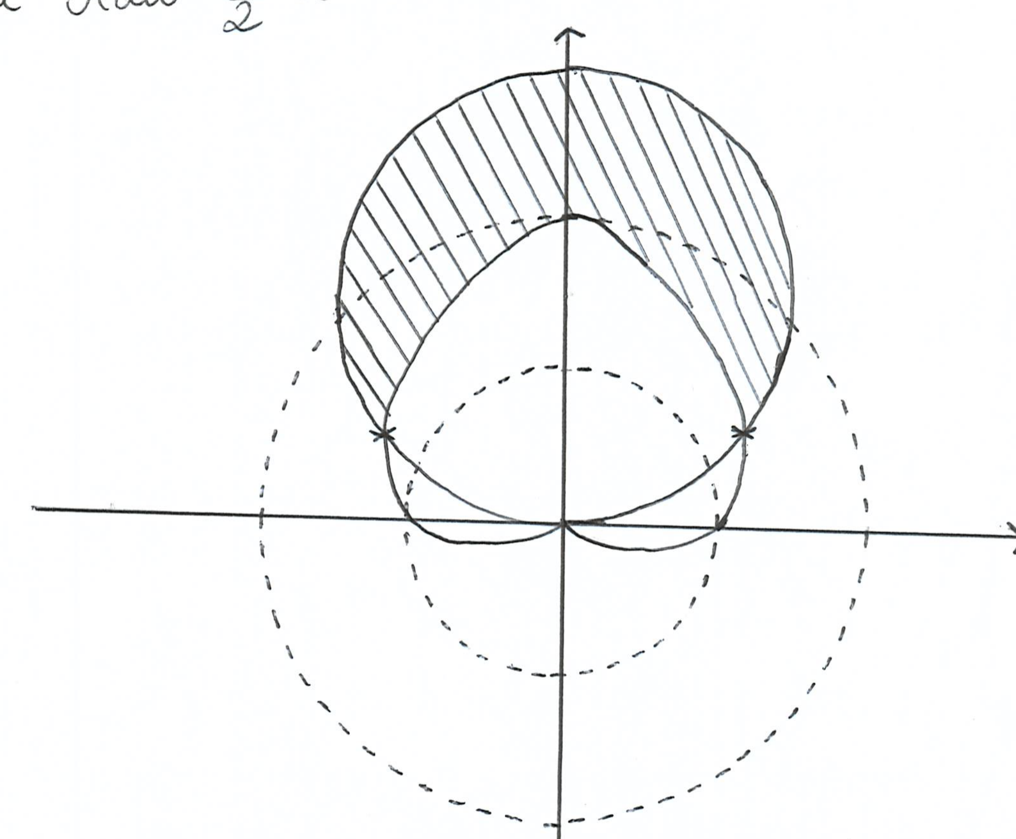
$$\vec{r}_1(t) = \left(\frac{3}{2}\,\text{sen}(2\theta);\ \frac{3}{2} - \frac{3}{2}\cos(2\theta)\right)$$

* Circunferência de raio $\frac{3}{2}$ e centro $\left(0, \frac{3}{2}\right)$

$r_2 = 1 + \text{sen}\,\theta$



θ	r
$0 - \frac{\pi}{2}$	$1 \nearrow 2$
$\frac{\pi}{2} - \pi$	$2 \searrow 1$
$\pi - \frac{3\pi}{2}$	$1 \searrow 0$
$\frac{3\pi}{2} - 2\pi$	$0 \nearrow 1$



$$1 + \text{sen}\,\theta = 3\,\text{sen}\,\theta \Rightarrow \theta \begin{cases} \frac{\pi}{6} \\ \frac{5\pi}{6} \end{cases}$$

$$A = \frac{1}{2}\int_{\frac{\pi}{6}}^{5\pi/6} (r_1^2 - r_2^2)\, d\theta = \frac{1}{2}\int_{\frac{\pi}{6}}^{5\pi/6} 9\,\text{sen}^2\theta - (1+\text{sen}\theta)^2\, d\theta$$

$$= \frac{1}{2}\int_{\frac{\pi}{6}}^{5\pi/6} 9\,\text{sen}^2\theta - 1 - 2\,\text{sen}\theta - \text{sen}^2\theta\, d\theta = \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} 8\,\text{sen}^2\theta - 2\,\text{sen}\theta - 1\, d\theta$$

$$* \ \frac{1}{2}\int 2\,\text{sen}\theta\, d\theta = -\cos\theta$$

$$* \ \frac{1}{2}\int 8\,\text{sen}^2\theta\, d\theta = 4\left(\frac{2\theta - \text{sen}(2\theta)}{4}\right) = 2\theta - \text{sen}(2\theta)$$

$$* \ \frac{1}{2}\int 1\, d\theta = \frac{\theta}{2}$$

$$\Rightarrow A = \left[2\theta - \text{sen}(2\theta) + \cos\theta - \frac{\theta}{2}\right]_{\pi/6}^{\frac{5\pi}{6}}$$

$$\Rightarrow A = \frac{10\pi}{6} - \frac{3\sqrt{3}}{6} + \frac{3\sqrt{3}}{6} - \frac{5\pi}{12} - \frac{2\pi}{6} + \frac{3\sqrt{3}}{6} - \frac{3\sqrt{3}}{6} + \frac{\pi}{12}$$

$$A = \frac{16\pi}{12} - \frac{4\pi}{12} = \frac{12\pi}{12} = \underline{\pi}$$

Questão 2 (3,0) Esboce as trajetórias das seguintes curvas parametrizadas.

(a) (1,0) $\alpha(t) = (e^{-t}, 1 + e^{2t})$, $t \geq 0$.

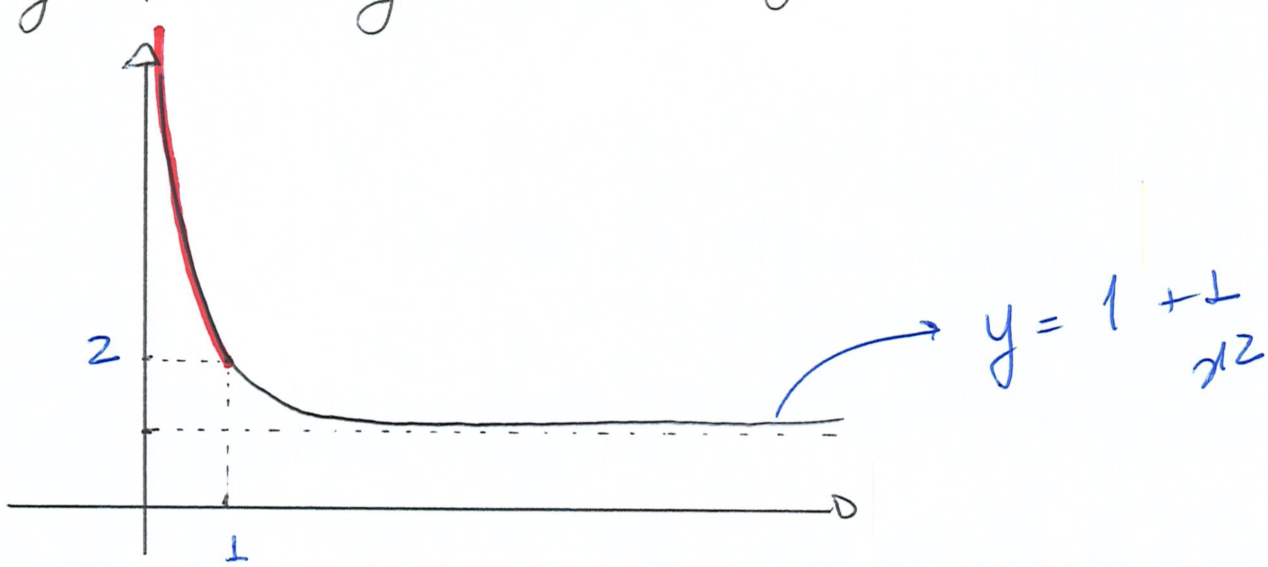
(b) (1,0) $\gamma(t) = (2 - \cos t, \sin^2 t + 3)$, $t \in [0, 2\pi]$.

(c) (1,0) $\beta(t) = (\tan t + \sec t, \tan t - \sec t)$, $t \in]0, \frac{\pi}{2}[$.

a $\alpha(t) = (e^{-t}; 1 + e^{2t})$

$x = e^{-t} \Rightarrow x = \frac{1}{e^t} \Rightarrow \frac{1}{x} = e^t$

$y = 1 + e^{2t} \Rightarrow y = 1 + (e^t)^2 \Rightarrow y = 1 + \left(\frac{1}{x}\right)^2 \Rightarrow y = 1 + \frac{1}{x^2}$



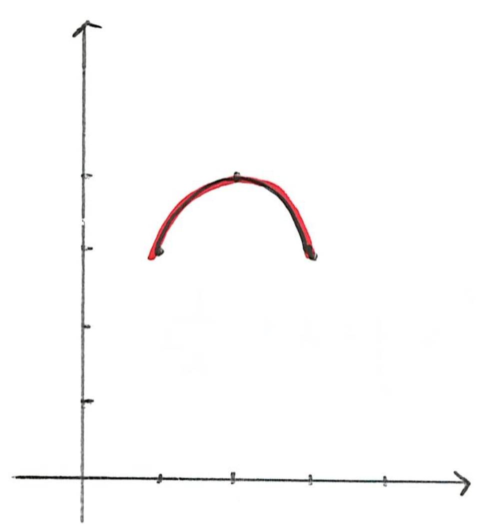
b) $(2-\cos t \,;\, \operatorname{sen}^2 t + 3)$ $t \in [0, 2\pi]$

$x = 2 - \cos t \Rightarrow \cos t = 2 - x$

$y = \operatorname{sen}^2 t + 3 \Rightarrow \operatorname{sen}^2 t = y - 3$

$* \; y - 3 + (2-x)^2 = 1 \Rightarrow y = 4 - (4 - 4x + x^2)$

$\Rightarrow y = 4x - x^2 \Rightarrow y = x(4-x)$

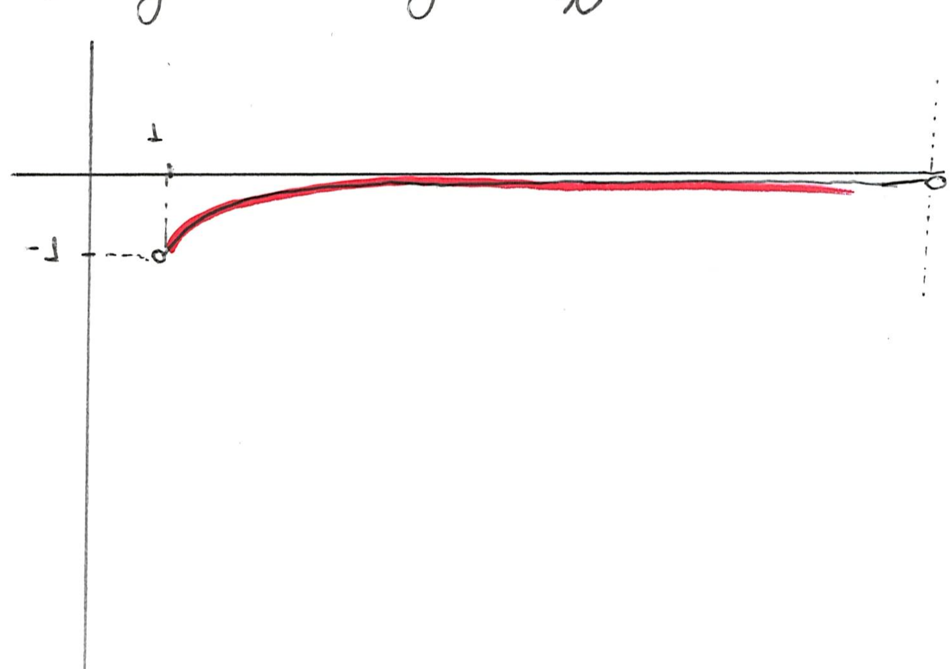


2c) $(\tan\theta + \sec\theta \,;\, \tan\theta - \sec\theta)$ $\theta \in]0; \frac{\pi}{2}[$

$x \cdot y = (\tan\theta + \sec\theta)(\tan\theta - \sec\theta) = \left(\frac{\operatorname{sen}\theta}{\cos\theta} + \frac{1}{\cos\theta}\right)\left(\frac{\operatorname{sen}\theta}{\cos\theta} - \frac{1}{\cos\theta}\right)$

$\Rightarrow xy = \frac{\operatorname{sen}^2\theta - 1}{\cos^2\theta} = -\frac{\cos^2\theta}{\cos^2\theta} = -1$

$\therefore xy = -1 \Rightarrow y = -\frac{1}{x}$



Questão 3 (2,5) Considere a curva $C(t) = (t^2, t^3 - 2t)$, $t \in \mathbb{R}$.

(a) (1,5) Em que ponto a curva intercepta ela mesma? Encontre as equações de ambas as retas tangentes neste ponto.

(b) (1,0) Em que pontos da curva a reta tangente é paralela à reta com equações $x = -1 + t$ e $y = 2t$? Encontre as equações destas retas tangentes.

(a) $C(t) = (t^2, t^3 - 2t)$; $C'(t) = (2t ; 3t^2 - 2)$

* A curva intercepta a si mesma no ponto $(2,0)$ para $t = \sqrt{2}$ e $t = -\sqrt{2}$

$R_{T,\sqrt{2}} = (2,0) + t(2\sqrt{2}, 4)$

$R_{T,-\sqrt{2}} = (2,0) + t(-2\sqrt{2}, 4)$

(b) $x' = 1$, $y' = 2$

$$\frac{dy}{dx} = 2 \Rightarrow 2 = \frac{3t^2 - 2}{2t} \Rightarrow 4t = 3t^2 - 2$$

$$\frac{4 \pm \sqrt{16 + 24}}{6} \begin{cases} \to \dfrac{2 + \sqrt{10}}{3} \\ \to \dfrac{2 - \sqrt{10}}{3} \end{cases}$$

Para $t = \frac{2 + \sqrt{10}}{3}$ temos $P\left(\left(\frac{2 + \sqrt{10}}{3}\right)^2 ; \left(\frac{2 + \sqrt{10}}{3}\right)^3 - 2\left(\frac{2 + \sqrt{10}}{3}\right)\right)$

$R_{T,\frac{2+\sqrt{10}}{3}} = \left(\left(\frac{2 + \sqrt{10}}{3}\right)^2 ; \left(\frac{2 + \sqrt{10}}{3}\right)^3 - 2\left(\frac{2 + \sqrt{10}}{3}\right)\right) + t\left(2\left(\frac{2 + \sqrt{10}}{3}\right) ; 3\left(\frac{2 + \sqrt{10}}{3}\right)^2 -\right.$

Para $t = \frac{2-\sqrt{10}}{3}$ temos:

$$P\left(\left(\frac{2-\sqrt{10}}{3}\right)^2; \left(\frac{2-\sqrt{10}}{3}\right)^3 - 2\left(\frac{2-\sqrt{10}}{3}\right)\right)$$

$$R_{t,\frac{2-\sqrt{10}}{3}} = \left(\left(\frac{2-\sqrt{10}}{3}\right)^2; \left(\frac{2-\sqrt{10}}{3}\right)^3 - 2\left(\frac{2-\sqrt{10}}{3}\right)\right) + t\left(2\left(\frac{2-\sqrt{10}}{3}\right); 3\left(\frac{2-\sqrt{10}}{3}\right)^2 - 2\right)$$

Questão 3 (2,5) Considere a curva $C(t) = (t^2, t^3 - 2t)$, $t \in \mathbb{R}$.

(a) (1,5) Em que ponto a curva intercepta ela mesma? Encontre as equações de ambas as retas tangentes neste ponto.

(b) (1,0) Em que pontos da curva a reta tangente é paralela à reta com equações $x = \cancel{1+t}$ $2t$ e $\cancel{y = 2t}$? Encontre as equações destas retas tangentes.

$y = -1 + t$

(a) $C(t) = (t^2 ; t^3 - 2t)$; $C'(t) = (2t ; 3t^2 - 2)$

* A curva intercepta a si mesma no ponto $(2,0)$ para $t = \sqrt{2}$ e $t = -\sqrt{2}$

$R_{T,\sqrt{2}} = (2,0) + t(2\sqrt{2}, 4)$

$R_{T,-\sqrt{2}} = (2,0) + t(-2\sqrt{2}, 4)$

(b) $x = 2t$ e $y = -1 + t \Rightarrow x' = 2$ e $y' = 1$

$$\frac{dy}{dx} = \frac{1}{2} \Rightarrow \frac{3t^2 - 2}{2t} = \frac{1}{2} \Rightarrow 6t^2 - 4 = 2t$$

$$\frac{1 \pm \sqrt{1+24}}{6} \begin{cases} \to \frac{6}{6} = 1 \\ \to -\frac{4}{6} = -\frac{2}{3} \end{cases}$$

Para $t = 1$ temos: $P(1, -1)$ e $R_{T,1} = (1, -1) + t(2, 1)$

Para $t = -\frac{2}{3}$ temos: $P\left(\frac{4}{9}, \frac{28}{27}\right)$ e $R_{T,-\frac{2}{3}} = \left(\frac{4}{9}, \frac{28}{27}\right) + t\left(-\frac{4}{3} ; -\frac{2}{3}\right)$

Questão 4 (2,0)

(a) (1,0) Determine o comprimento da curva $x(t) = e^t - t$, $y(t) = 4e^{\frac{t}{2}}$, para $0 \leq t \leq 2$.

(b) (1,0) Seja $C : A \subset \mathbb{R} \longrightarrow \mathbb{R}^2$ uma curva com componentes $x(t)$ e $y(t)$. Dizemos que a curva C é **regular** se $C'(t) \neq \vec{0}$, para todo $t \in A$. Para qualquer curva regular, podemos definir sua **curvatura no ponto t** através da expressão:

$$\kappa(t) = \frac{|x'(t)y''(t) - x''(t)y'(t)|}{(x'(t)^2 + y'(t)^2)^{\frac{3}{2}}}.$$

Mostre que a curvatura em todo ponto de qualquer reta é zero.

(a) $$l = \int_0^2 \sqrt{(e^t - 1)^2 + (2e^{t/2})^2}\, dt = \int_0^2 \sqrt{e^{2t} - 2e^t + 1 + 4e^t}\, dt$$

$$= \int_0^2 \sqrt{(e^t)^2 + 2e^t + 1}\, dt = \int_0^2 \sqrt{(e^t + 1)^2}\, dt = \int_0^2 |e^t + 1|\, dt$$

* Como $t \in [0, 2]$ temos que

$$l = \int_0^2 e^t + 1\, dt = \left[e^t + t\right]_0^2 = e^2 + 2 - e^0 = \underline{e^2 + 1}$$

(b) $\vec{R}(t) = (f(t), g(t)) \Rightarrow \vec{R}(t) = (at+b, ct+d)$

$x'(t) = a \quad , \quad x''(t) = 0$

$y'(t) = c \quad , \quad y''(t) = 0$

$$\Rightarrow K(t) = \frac{a \cdot 0 + 0 \cdot c}{(a^2 + b^2)^{3/2}} = 0 \quad \forall t \in R$$