(a) *Encontre a área da região do plano cartesiano limitada pela parábola $y = x^2$, pela reta tangente a esta parábola em $(1, 1)$ e pelo eixo x.*

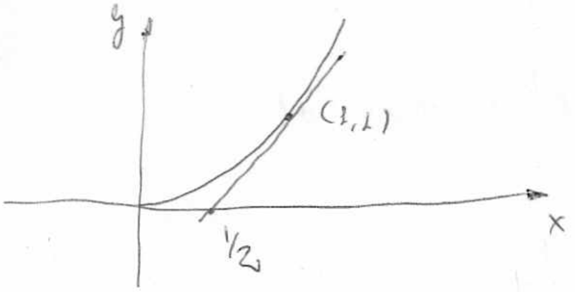
(b) *Calcule* $\int (\ln x)^2 \, dx$.

(c) *Calcule* $\int \sqrt{x^2 - 2x + 2} \, dx$.

(a) $f : \mathbb{R} \to \mathbb{R}$, $x \mapsto x^2$ $\qquad f' : \mathbb{R} \to \mathbb{R}$, $x \mapsto 2x$ $\quad \therefore f'(1) = 2$

$\therefore$ a reta tangente à parábola $y = x^2$ em $(1,1)$ tem equação $y - 1 = 2(x-1)$

Esta reta intersecta o eixo Ox no ponto $(1/2, 0)$.



Assim, a área pedida é dada por:

$$A = \int_0^{1/2} x^2 \, dx + \int_{1/2}^1 (x^2 - [2x-1]) \, dx =$$

$$= \int_0^1 x^2 \, dx - \int_{1/2}^1 (2x-1) \, dx \overset{TFC}{=} \left[\frac{x^3}{3}\right]_0^1 - \left[x^2 - x\right]_{1/2}^1 =$$

$$= 1/3 + \frac{1}{4} - \frac{1}{2} = \boxed{\frac{1}{12}}$$

(b) $\int (\ln x)^2 \, dx = uv - \int v \, du = x(\ln x)^2 - 2\int \ln x \, dx$ (*)

$\begin{cases} u = (\ln x)^2, & du = \dfrac{2 \ln x}{x} \, dx \\ dv = dx, & v = x \end{cases}$

$$\int \underbrace{(\ln x)}_{u} \underbrace{dx}_{dv} = x \ln x - \int x \cdot \frac{1}{x} \, dx = x \ln x - x + K$$

$$\therefore (*) = \boxed{x(\ln x)^2 - 2x \ln x + 2x + K}$$

(c) $\int \sqrt{x^2 - 2x + 2}\, dx$

$$x^2 - 2x + 2 = (x^2 - 2x + 1) + 1 = (x-1)^2 + 1$$

Pomos $\begin{cases} x - 1 = tg\, u, \quad -\pi/2 < u < \pi/2 \iff u = arctg\,(x-1) \\ dx = sec^2 u\, du \\ \sqrt{x^2 - 2x + 2} = \sqrt{(x-1)^2 + 1} = \sqrt{tg^2 u + 1} = \\ \qquad = sec\, u \end{cases}$

$$\therefore \int \sqrt{x^2 - 2x + 2}\, dx = \int sec^3 u\, du \quad (*)$$

$\begin{cases} w = sec\, u, & dw = sec\, u\, tg\, u\, du \\ dv = sec^2 u\, du, & v = tg\, u \end{cases}$

$$(*) = wv - \int v\, dw = sec\, u\, tg\, u - \int sec\, u\, \overbrace{tg^2 u}^{= sec^2 u - 1}\, du =$$

$$= sec\, u\, tg\, u - \int sec^3 u\, du + \underbrace{\int sec\, u\, du}_{= \ln |sec\, u + tg\, u| + K}$$

$$\therefore (*) = \frac{1}{2} sec\, u\, tg\, u + \frac{1}{2} \ln |sec\, u + tg\, u| + K =$$

$$= \boxed{\frac{1}{2}(x-1)\sqrt{1 + (x-1)^2} + \frac{1}{2} \ln \left[\sqrt{1 + (x-1)^2} + (x-1) \right] + K}$$

QUESTÃO 2. (2 ptos.) *Dados $a, b > 0$, calcule a área da região do plano cartesiano limitada pela elipse* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

$$A = 2\int_{-a}^{a} b\sqrt{1-\left(\frac{x}{a}\right)^2}\,dx \quad (*)$$

Tome $g : [-\frac{\pi}{2}, \pi/2] \longrightarrow \mathbb{R}$, $t \longmapsto a\,\text{sen}\,t$. Então $g \in C^1$,

$g(-\frac{\pi}{2}) = -a$ e $g(\frac{\pi}{2}) = a$. Pelo TMV:

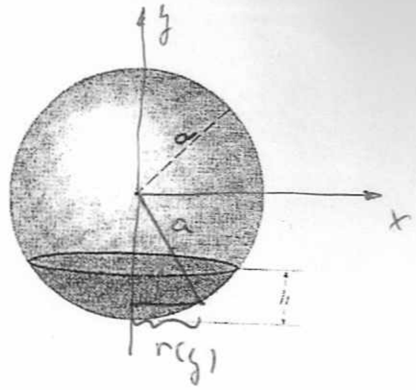
$$(*) = 2b\int_{-\pi/2}^{\pi/2} \sqrt{1-\text{sen}^2 t}\,.\,a\cos t\,dt =$$

$$= 2ab\int_{-\pi/2}^{\pi/2} \cos^2 t\,dt = 2ab\int_{-\pi/2}^{\pi/2} \frac{\cos 2t + 1}{2}\,dt \overset{\text{TFC}}{=}$$

$$= 2ab\left\{ \underbrace{\left[\frac{1}{2}\,\frac{\text{sen}\,2t}{2}\right]_{-\pi/2}^{\pi/2}}_{=0} + \left[\frac{1}{2}t\right]_{-\pi/2}^{\pi/2} \right\} = 2ab\,.\,\frac{\pi}{2} =$$

$$= \boxed{\pi a b}$$

QUESTÃO 3. (2 ptos.) *Dados números reais $0 < h < a$, calcule o volume de uma calota esférica de altura h de uma esfera de raio a.*



$$r(y)^2 = a^2 - y^2$$

$$V = \int_{-a}^{-a+h} \pi r(y)^2 \, dy = \pi \int_{-a}^{-a+h} (a^2 - y^2)\, dy \overset{TFC}{=}$$

$$= \pi \left[a^2 y - \frac{y^3}{3} \right]_{-a}^{-a+h} = \pi \left[-a^3 + a^2 h - \frac{(-a+h)^3}{3} - \right.$$

$$\left. - (-a^3) + \frac{(-a)^3}{3} \right] = \pi \left[-\cancel{a^3} + a^2 h - \frac{h^3 - 3h^2 a + 3ha^2 - a^3}{3} + \right.$$

$$\left. + \cancel{a^3} - \frac{a^3}{3} \right] = \frac{\pi}{3} \left[3\cancel{a^2 h} - h^3 + 3ah^2 - 3\cancel{a^2 h} + \cancel{a^3} - \cancel{a^3} \right] =$$

$$= \boxed{\frac{\pi}{3} h^2 (3a - h)}$$

$x_0 = 0$, *e use-o para calcular e com erro inferior a* 10^{-5}.

(b) *Mostre que, para todo* $n \in \mathbb{N}$:

$$\left| e^{x^2} - \sum_{k=0}^{n} \frac{x^{2k}}{k!} \right| \leqslant \frac{e^{x^2} x^{2n+2}}{(n+1)!}$$

(c) *Calcule* $\int_0^1 e^{x^2} dx$ *com erro inferior a* 10^{-5}.

(a) $f: \mathbb{R} \longrightarrow \mathbb{R}$, $x \longmapsto e^x$ $\quad \forall n \in \mathbb{N}$, $f^{(n)} = \exp$

$\therefore f^{(n)}(0) = 1$

$\therefore P_n(x) = \sum_{k=0}^{n} \frac{x^k}{k!}$

Pela fórmula de Taylor c/ resto de Lagrange, dado $x > 0$, $\exists\, c \in]0, x[\;/\; f(x) - P_n(x) = \frac{f^{(n+1)}(c)\, x^{n+1}}{(n+1)!} \overset{(*)}{=} \frac{e^c \cdot x^{n+1}}{(n+1)!}$. Em particular, p/ $x = 1$, $\exists\, c \in]0, 1[\;/$

$e - P_n(1) = \frac{e^c}{(n+1)!}$, donde $|e - P_n(1)| < \frac{3}{(n+1)!}$

Assim, se $\frac{3}{(n+1)!} < 10^{-5} \Leftrightarrow (n+1)! > 3 \cdot 10^5 \Leftrightarrow (n+1) \geqslant 9 \Leftrightarrow n \geqslant 8$,

obtém-se $|e - P_n(1)| < 10^{-5}$. #

(b) De $(*)$, no item anterior, segue-se que $\forall x > 0$:

$|f(x) - P_n(x)| \overset{(**)}{\leq} \frac{e^x \cdot x^{n+1}}{(n+1)!}$

Assim, $\forall x \in \mathbb{R} \setminus \{0\}$, $x^2 > 0$ $\therefore$ $(**)$ vale com x^2 no lugar de x, e vale a igualdade p/ $x = 0$. #

(c)

$$\left| \int_0^1 e^{x^2} dx - \int_0^1 P_n(x^2)\, dx \right| = \left| \int_0^1 \left[e^{x^2} - P_n(x^2) \right] dx \right| \leq$$

$$\leq \int_0^1 \left| e^{x^2} - P_n(x^2) \right| dx \overset{\text{item anterior}}{\leq} \int_0^1 \frac{e^{x^2} \cdot x^{2n+2}}{(n+1)!}\, dx \leq$$

$$\leq 3 \int_0^1 \frac{x^{2n+2}}{(n+1)!}\, dx \overset{TFC}{=} \frac{3}{(n+1)!} \left. \frac{x^{2n+3}}{2n+3} \right]_0^1 = \frac{3}{(2n+3)(n+1)!}$$

Assim, se $\dfrac{3}{(2n+3)(n+1)!} < 10^{-5} \Leftrightarrow (2n+3)(n+1)! > 3 \cdot 10^5 \Leftrightarrow$

$\Leftrightarrow n+1 \geq 8 \Leftrightarrow n \geq 7$, tem-se

$$\left| \int_0^1 e^{x^2} dx - \int_0^1 P_n(x^2)\, dx \right| < 10^{-5}.$$

Assim, $\displaystyle\int_0^1 P_7(x^2)\, dx = \int_0^1 \sum_{k=0}^{7} \frac{x^{2k}}{k!}\, dx \overset{TFC}{=}$

$= \displaystyle\sum_{k=0}^{7} \frac{1}{k!} \left. \frac{x^{2k+1}}{2k+1} \right]_0^1 = \sum_{k=0}^{7} \frac{1}{(2k+1) \cdot k!}$ aproxima $\displaystyle\int_0^1 e^{x^2} dx$

com erro inferior a 10^{-5}. #