Questão 1) (1.0) (a) Utilizando o polinômio de Taylor de ordem 3, da função $f(x) = \sqrt[3]{1+x}$ em torno de $x_o = 0$, calcule um valor aproximado para $\sqrt[3]{1,2}$.

(0.5) (b) Pode-se afirmar que, em módulo o erro cometido em (a) é inferior a $\frac{2^4.10^{-3}}{3^5}$? **Justifique.**

$$f(x) = (1+x)^{1/3}$$

$$f'(x) = \frac{1}{3}(1+x)^{-2/3}$$

$$f''(x) = -\frac{2}{3^2}(1+x)^{-5/3}$$

$$f'''(x) = \frac{10}{3^3}(1+x)^{-8/3}$$

$$f^{(4)}(x) = -\frac{80}{3^4}(1+x)^{-11/3}$$

$$P_{3,0}(x) = f(0) + f'(0)(x-0) + f''(0)\frac{(x-0)^2}{2!} + \frac{f'''(0)(x-0)^3}{3!} = 1 + \frac{1}{3}x - \frac{2}{3^2 . 2}x^2 + \frac{10}{3^3 3!}x^3$$

$$= 1 + \frac{x}{3} - \frac{1}{9}x^2 + \frac{5}{81}x^3$$

a) $$\sqrt[3]{1,2} \simeq P_{3,0}(0,2) = 1 + \frac{0,2}{3} - \frac{1}{9}(0,2)^2 + \frac{5}{81}(0,2)^3$$

b) $$|\text{Erro}| = \left| \frac{f^{(4)}(c)\,(0,2-0)^4}{4!} \right|$$ para algum $0 < c < 0,2$

$$|\text{Erro}| = \frac{2^4 . 10^{-4}}{2.3.4}\,\frac{80}{3^4}\,\frac{1}{\sqrt[3]{(1+c)^{11}}} = \frac{2^4\,10^{-3}}{3^5}\,\frac{1}{\sqrt[3]{(1+c)^{11}}}$$

Mas sabemos que $0 < c < 0,2$ e assim $\sqrt[3]{(1+c)^{11}} > 1$. Portanto $0 < \frac{1}{\sqrt[3]{(1+c)^{11}}} < 1$;

e assim $$|\text{Erro}| = \frac{2^4 . 10^{-3}}{3^5}\,\frac{1}{\sqrt[3]{(1+c)^{11}}} < \frac{2^4 . 10^{-3}}{3^5}$$

Questão 2) Determine todas as soluções $y = y(x)$ das equações:

(1.0) (a) $y' = \dfrac{xe^{2x}}{(1+3y)^2}$ (1.0) (b) $y' + \dfrac{4x^3}{5+x^4}\, y = x$ (1.0) (c) $y'' - y' - 6y = e^{3x}$

a) Variáveis separáveis

$$\frac{dy}{dx} = \frac{xe^{2x}}{(1+3y)^2}$$

$$(1+3y)^2 dy = xe^{2x} dx$$

$$\int (1+3y)^2 dy = \frac{(1+3y)^3}{3}\cdot\frac{1}{3} + k_1$$

$$\int \underbrace{x}_{f}\underbrace{e^{2x}}_{g'} dx = \frac{x.\,e^{2x}}{2} - \int \frac{e^{2x}}{2} dx =$$

$$= \frac{xe^{2x}}{2} - \frac{1}{2}\frac{e^{2x}}{2} + k_2$$

Portanto

$$\frac{(1+3y)^3}{9} = \frac{xe^{2x}}{2} - \frac{e^{2x}}{4} + k$$

$$(1+3y)^3 = 9\left[\frac{xe^{2x}}{2} - \frac{e^{2x}}{4} + k\right]$$

$$1+3y = \sqrt[3]{9\left(\frac{xe^{2x}}{2} - \frac{e^{2x}}{4} + k\right)}$$

$$y = \frac{1}{3}\left[\sqrt[3]{9\left(\frac{xe^{2x}}{2} - \frac{e^{2x}}{4} + k\right)} - 1\right],$$

$k \in \mathbb{R}$

b) linear de 1ª ordem com coeficientes não constantes

$$g(x) = \frac{4x^3}{5+x^4}$$

G primitiva de g

$$G(x) = \ln(5+x^4)$$

$$\left(e^{G(x)}y\right)' = e^{G(x)}x$$

$$e^{G(x)} = e^{\ln(5+x^4)} = 5+x^4$$

$$\left((5+x^4)y\right)' = (5+x^4)x$$

$$(5+x^4)y = \int (5x+x^5)\,dx$$

$$(5+x^4)y = \frac{5x^2}{2} + \frac{x^6}{6} + k$$

$$y = \frac{1}{5+x^4}\left(\frac{5x^2}{2} + \frac{x^6}{6} + k\right),$$

com $k \in \mathbb{R}$

$$y = \frac{\int (5+x^4)\ldots}{5+x^4}$$

$$\int \frac{4x^3}{5+x^4} = \ln|5+x^4|$$

$$(e\,y)' = e\,x \qquad y = \frac{5x+x^5}{5+x^4} + k$$

c) linear de 2ª ordem com coeficientes constantes

$$y'' - y' - 6y = 0$$

$$\beta^2 - \beta - 6 = 0$$

$$\Delta = 1 + 24 = 25$$

$$\beta = \frac{1 \pm 5}{2} \to 3,\ -2$$

$$y_H = Ae^{-2x} + Be^{3x}$$

$$y_p = Cxe^{3x}$$

$$y_p' = C\left[e^{3x} + xe^{3x}3\right]$$

$$y_p'' = C\left[e^{3x}3 + 3e^{3x} + 9xe^{3x}\right]$$

$$y_p'' - y_p' - 6y_p = e^{3x}$$

$$e^{3x}(6C - C) + xe^{3x}(9C - 3C - 6C) = e^{3x}$$

Portanto $5C = 1$

$C = 1/5$

$$y = Ae^{-2x} + Be^{3x} + C\frac{1}{5}xe^{3x}$$

sendo $A, B, C \in \mathbb{R}$.

Questão 3) Calcule

(1.0) (a) $\int \frac{\sqrt[5]{8+\sqrt{x}}}{3\sqrt{x}}\,dx$ (1.5)(b) $\int \frac{\sqrt{9+4x^2}}{x^4}\,dx$.

a) $s = 8+\sqrt{x}$

$ds = \frac{1}{2\sqrt{x}}\,dx$

$$\int \frac{\sqrt[5]{8+\sqrt{x}}}{3\sqrt{x}}\,dx = \int \frac{2}{3}\sqrt[5]{8+\sqrt{x}}\ \frac{1}{2\sqrt{x}}\,dx$$

$$\int \frac{2}{3}\sqrt[5]{s}\,ds = \frac{2}{3}\,\frac{s^{1/5+1}}{\frac{1}{5}+1} + k = \frac{\cancel{2}\cdot 5}{3\cdot\cancel{6}\,3}\sqrt[5]{s^6} + k$$

$$\int \frac{\sqrt[5]{8+\sqrt{x}}}{3\sqrt{x}}\,dx = \frac{5}{9}\sqrt[5]{\left(8+\sqrt{x}\right)^6} + k,\quad k\in\mathbb{R}$$

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b) $$\int \frac{\sqrt{9+4x^2}}{x^4}\,dx = \int \frac{3\sqrt{1+\left(\frac{2x}{3}\right)^2}}{x^4}\,dx$$

$\frac{2x}{3} = tg\, s\ ,\ -\pi/2 < s < \pi/2$

$x = \frac{3}{2}\,tg\, s \qquad dx = \frac{3}{2}\sec^2 s\,ds$

$$\int \frac{3\sqrt{1+tg^2 s}}{\left(\frac{3}{2}\right)^4 tg^4 s}\,\frac{3}{2}\sec^2 s\,ds = \int \frac{8}{9}\,\frac{\sqrt{\sec^2 s}\ \sec^2 s}{tg^4 s}\,ds = \int \frac{8}{9}\,\frac{|\sec s|\sec^2 s}{tg^4 s}\,ds$$

$\sec s > 0$ pois $s \in \left]-\pi/2, \pi/2\right[$

$$= \int \frac{8}{9}\,\frac{\sec^3 s}{tg^4 s}\,ds = \int \frac{8}{9}\,\frac{1}{\cos^3 s}\,\frac{\cos^4 s}{sen^4 s}\,ds = \frac{8}{9}\int \frac{\cos s}{sen^4 s}\,ds = (*)$$

$t = sen\, s$

$dt = \cos s\,ds$

$$\int \frac{1}{t^4}\,dt = \frac{t^{-4+1}}{-3} + k_1 = -\frac{1}{3}\,\frac{1}{t^3} + k_1$$

$$(*) = \frac{8}{9}\left(-\frac{1}{3}\right)\frac{1}{(sen\, s)^3} + k_1 = -\frac{8}{27}\left(\frac{\sec s}{tg\, s}\right)^3 + k_1 = -\frac{8}{27}\left(\frac{\sqrt{1+tg^2 s}}{tg\, s}\right)^3 + k_1$$

Portanto

$$\int \frac{\sqrt{9+4x^2}}{x^4}\,dx = -\frac{8}{27}\left(\frac{\sqrt{1+\left(\frac{2x}{3}\right)^2}}{\frac{2x}{3}}\right)^3 + k_1$$

Questão 4) Calcule

(1.5) (a) $\int x^2 arctg(\sqrt{2x+4})\, dx$ (1.5) (b) $\int \frac{9x+5}{x(x^2+4x+5)}$

$s = \sqrt{2x+4} \qquad s^2 = 2x+4 \qquad x = \frac{s^2-4}{2}$

$ds = \frac{1}{\cancel{2}\sqrt{2x+4}} \cancel{2}\, dx$

$\int x^2 \operatorname{arctg}(\sqrt{2x+4})\,dx = \int x^2 \sqrt{2x+4}\, \operatorname{arctg}(\sqrt{2x+4})\, \frac{1}{\sqrt{2x+4}}\,dx$

$\int \left(\frac{s^2-4}{2}\right)^2 s \operatorname{arctg} s\, ds = \int \frac{1}{4}(s^4 - 8s^2 + 16)\, s \operatorname{arctg} s\, ds = \frac{1}{4}\int \underbrace{(s^5 - 8s^3 + 16s)}_{f'} \underbrace{\operatorname{arctg} s}_{g}$

$= \frac{1}{4}\left[\left(\frac{s^6}{6} - \frac{8s^4}{4} + \frac{16s^2}{2}\right)\operatorname{arctg} s - \int \left(\frac{s^6}{6} - \frac{8s^4}{4} + \frac{16s^2}{2}\right)\frac{1}{1+s^2}\,ds\right]$

$\int \left(\frac{s^6}{6} - \frac{\cancel{8}^{2} s^4}{\cancel{4}} + \frac{\cancel{16}^{8} s^2}{\cancel{2}}\right)\frac{1}{1+s^2}\,ds = \int \frac{s^6 - 12s^4 + 48s^2}{6}\,\frac{1}{1+s^2}\,ds =$

$= \frac{1}{6}\int \frac{(s^2+1)(s^4 - 13s^2 + 61) - 61}{1+s^2}\,ds = 1/6 \int (s^4 - 13s^2 + 61)\,ds - \frac{61}{6}\int \frac{1}{1+s^2}\,ds =$

$= \frac{1}{6}\left[\frac{s^5}{5} - \frac{13s^3}{3} + 61s - 61\operatorname{arctg} s\right] + k$

Portanto

$\int \left(\frac{s^2-4}{2}\right)^2 s \operatorname{arctg} s\, ds = \frac{1}{4}\left[\left(\frac{s^6}{6} - 2s^4 + 8s^2\right)\operatorname{arctg} s - \frac{1}{6}\left(\frac{s^5}{5} - \frac{13s^3}{3} + 61s - 61\operatorname{arctg} s\right)\right] +$

$\int x^2 \operatorname{arctg}(\sqrt{2x+4})\,dx = \frac{1}{4}\left[\left(\frac{(\sqrt{2x+4})^6}{6} - 2(\sqrt{2x+4})^4 + 8(2x+4)\right)\operatorname{arctg}\sqrt{2x+4}\right] +$

$-\frac{1}{24}\left(\frac{(\sqrt{2x+4})^5}{5} - 13\frac{(\sqrt{2x+4})^3}{3} + 61\sqrt{2x+4} - 61\operatorname{arctg}\sqrt{2x+4}\right) + k, \quad k \in \mathbb{R}$

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b) $\frac{9x+5}{x(x^2+4x+5)} = \frac{A}{x} + \frac{Bx+C}{x^2+4x+5}$

$A + B = 0$

$4A + C = 9$

$5A = 5$

$A = 1, \quad B = -1, \quad C = 5$

$\overset{\shortparallel}{\frac{A(x^2+4x+5) + x(Bx+C)}{x(x^2+4x+5)}}$

$\frac{9x+5}{x(x^2+4x+5)} = \frac{1}{x} + \frac{5-x}{x^2+4x+5}$

$\int \frac{5-x}{x^2+4x+5}\,dx = \int \frac{-\frac{1}{2}(2x+4) + 7}{x^2+4x+5}\,dx = -\frac{1}{2}\ln(x^2+4x+5) +$

$7\operatorname{arctg}(x+2) + k$

$+\int \frac{7}{(x+2)^2+1}\,dx = -1/2 \ln(x^2+4x+5) + 7\operatorname{arctg}(x+2) + k,$

Portanto $\int \frac{9x+5}{x(x^2+4x+5)}\,dx = \ln|x| - \frac{1}{2}\ln(x^2+4x+5) + 7\operatorname{arctg}(x+2) + k,$

$k \in \mathbb{R}$